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The Bunce-Deddens Algebras as Crossed Products by Partial Automorphisms

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Abstract. We describe both the Bunce-Deddens C^*-algebras and their Toeplitz versions, as crossed products of commutative C^*-algebras by partial automorphisms. In the latter case, the commutative algebra has, as its spectrum, the union of the Cantor set and a copy of the set of natural numbers $\mathbb{N}$, fitted together in such a way that $\mathbb{N}$ is an open dense subset. The partial automorphism is induced by a map that acts like the odometer map on the Cantor set while being the translation by one on $\mathbb{N}$. From this we deduce, by taking quotients, that the Bunce-Deddens C^*-algebras are isomorphic to the (classical) crossed product of the algebra of continuous functions on the Cantor set by the odometer map.

1. Introduction

Recall from [4] that a weighted shift operator is a bounded operator on $l_2 = l_2(\mathbb{N})$ given, on the canonical basis $\{e_n\}_{n=0}^{\infty}$, by $S_a(e_n) = a_{n+1}e_{n+1}$, where the weight sequence $a = \{a_n\}_{n=1}^{\infty}$, is a bounded sequence of complex numbers. A weighted shift is said to be p-periodic if its weights satisfy $a_n = a_{n+p}$ for all n.

Given a strictly increasing sequence $\{n_k\}_{k=0}^{\infty}$ of positive integers, such that n_k divides n_{k+1} for all k, the Bunce-Deddens-Toeplitz C^*-algebra $A = A(\{n_k\})$ is defined to be the C^*-algebra of operators on l_2, generated by the set of all n_k-periodic weighted shifts, for all k. These algebras were first studied by Bunce and Deddens in [5]. It was observed by them that the algebra $\mathcal{K}$, of compact operators on l_2, is contained in A and that the quotient $A/\mathcal{K}$ is a simple C^*-algebra. The latter became known as the Bunce-Deddens C^*-algebra and has been extensively studied (see, for example, [1], [2, 10.11.4], [3], [6], [9, p.

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248], [**11**], [**12**], [**13**]). We shall denote these algebras by $B(\{n_k\})$ or simply by B, if the weight sequence is understood.

The goal of the present work is to describe both $A(\{n_k\})$ and $B(\{n_k\})$ as the crossed product of commutative C^*-algebras by partial automorphisms [**7**], in much the same way as we have described general AF-algebras [**8**] as partial crossed products.

In the case of $A(\{n_k\})$, we shall see that it is given by a curious (partial) dynamical system consisting of a topological space X which can be thought of as a compactification of the (discrete) space $\mathbb{N}$ of natural numbers, the complement of $\mathbb{N}$ in X being homeomorphic to the Cantor set $\mathbb{K}$. The transformation f of X, by which the partial crossed product is taken, leaves both $\mathbb{N}$ and $\mathbb{K}$ invariant. Its behavior on $\mathbb{N}$ is that of the translation by one, while the action on $\mathbb{K}$ is by means of the odometer map (see, for example, [**12**]) which is defined as follows. Given a sequence $\{q_k\}_{k=0}^{\infty}$ of positive integers (below we shall use $q_k = n_{k+1}/n_k$), consider the Cantor set, as given by the model

$$\mathbb{K} = \prod_{j=0}^{\infty} \{0, 1, \dots, q_j - 1\}.$$

The odometer map is the map $\mathcal{O}: \mathbb{K} \to \mathbb{K}$, given by formal addition of $(1, 0, \dots)$ with carry over to the right. Note that

$$\mathcal{O}(q_0 - 1, q_1 - 1, \dots) = (0, 0, \dots)$$

since the carry over process, in this case, extends all the way to infinite. For further reference let us call by the name of "partial odometer" the restriction of $\mathcal{O}$ to a map from

$$X - \{(q_0 - 1, q_1 - 1, \dots)\} \quad \text{to} \quad X - \{(0, 0, \dots)\}$$

so that, for this map, the carry over process always terminates in finite time.

As already mentioned, $B(\{n_k\})$ is the quotient of $A(\{n_k\})$ by $\mathcal{K}$. But $\mathcal{K}$ can be seen to correspond to the restriction of the above dynamical system to $\mathbb{N}$ (see [**7**]). So, we deduce that the Bunce-Deddens algebras $B(\{n_k\})$ are isomorphic to the crossed product of the Cantor set by the odometer map. This result is already well known [**2**, 10.11.4] but it is interesting to remark how little bookkeeping is necessary to deduce it from the machinery of partial automorphisms [**7**]. Moreover, this should be compared with [**8**], Theorem 3.2, according to which UHF-algebras sit as subalgebras of $B(\{n_k\})$ (see also [**12**]).

This work was done while I was visiting the Mathematics Department at the University of New Mexico.

2. Circle Actions

For each z in the unit circle

$$S^1 = \{w \in \mathbb{C}: |w| = 1\}$$

let U_z denote the diagonal unitary operator on l_2, given by $U_z(e_n) = z^n e_n$. If S is any weighted shift, it is easy to see that $U_z S U_z^{-1} = zS$. Thus, denoting by α_z the inner automorphism of $\mathcal{B}(l_2)$ given by conjugation by U_z, one finds that A is invariant under α_z. Moreover, one can see that this gives a continuous action of S^1 on A, in the sense of [**10**], 7.4.1 (even though the corresponding action is not continuous on $\mathcal{B}(l_2)$).

Let us denote the fixed point subalgebra by A_0. It is easy to see that A_0 consists precisely of the operators in A which are diagonal with respect to the basis $\{e_n\}$. Now, given that the C^*-algebra of (bounded) diagonal operators is isomorphic to $l_\infty = l_\infty(\mathbb{N})$, it is convenient to view A_0 as a subalgebra of l_∞. For the purpose of describing A_0, observe that, since $\mathcal{K}$ is contained in A, it follows that c_0 (the subalgebra of l_∞ formed by sequences tending to zero) is contained in A_0. Carrying this analysis a bit further one can prove that $A_0 = c_0 \oplus D$ where D is the subalgebra of l_∞ generated by all n_k-periodic sequences, for all k.

This decomposition is useful in determining the spectrum of A_0. Note, initially, that the spectrum of c_0 is homeomorphic to $\mathbb{N}$ (with the discrete topology) while the spectrum of D is the Cantor set, here denoted $\mathbb{K}$. This said, one has that the spectrum of A_0 can be described, at least in set theoretical terms, as the union $X = \mathbb{N} \cup \mathbb{K}$. Moreover, since c_0 is an essential ideal in A_0, one sees that $\mathbb{N}$ is an open dense subset of X. To better grasp the entire topology of X we need a more precise notation. Assume, without loss of generality, that $n_0 = 1$ and let $q_k = n_{k+1}/n_k$ for $k \geq 0$. Any integer n with $0 \leq n < n_k$ has a unique representation as

$$n = \sum_{j=0}^{k-1} \beta_j^{(n)} n_j$$

where $0 \leq \beta_j^{(n)} < q_j$. Here the $\beta_j^{(n)}$ play the role of digits in a decimal-like representation, except that the base varies along with the position of each digit. Accordingly, we let $\beta^{(n)} = (\beta_0^{(n)}, \ldots, \beta_{k-1}^{(n)})$ be the corresponding notation for n (which we shall use interchangeably without further warning). When convenient, we shall also view $\beta^{(n)}$ as an element of the set

$$\mathbb{K}_k = \prod_{j=0}^{k-1} \quad \{0, 1, \ldots, q_j - 1\}.$$

For each k and each β in $\mathbb{K}_k$, we denote by e_β the n_k-periodic sequence (thus an element of D) given by

$$e_\beta(n) = \begin{cases} 1 & \text{if} \quad n \equiv \beta \pmod{n_k} \\ 0 & \text{otherwise} \end{cases}$$

Note that the length of β determines which n_k should be used in the above definition. Clearly the set $\{e_\beta \in \cup_{k=1}^{\infty} \mathbb{K}_k\}$ generates D. Making use of the notation introduced, we shall adopt for the Cantor set, the model

$$\mathbb{K} = \prod_{j=0}^{\infty} \quad \{0, 1, \ldots, q_j - 1\}$$

so that, once we view D as the algebra of continuous functions on $\mathbb{K}$ via the Gelfand transform, the support of the e_β form a basis for the topology of $\mathbb{K}$. In fact, the support of e_β is precisely the set of elements $\gamma = (\gamma_i)$ in $\mathbb{K}$ such that $\gamma_j = \beta_j$ for all $j = 0, \ldots, k-1$ (assuming that β is in $\mathbb{K}_k$). That is, γ is in the support of β if and only if its initial segment coincides with β.

Considering now, the whole of A_0, note that it is generated by the set of all idempotents p which, viewed as elements of l_∞, have one of the two following forms: either it has a finite number of non-zero coordinates (in which case p is in c_0), or it coincide with some e_β, except for finitely many coordinates. The set of such idempotents is closed under multiplication, which therefore implies that their support, in the spectrum X of A_0, form a basis for the topology of X. With this we have precisely described the topology of X:

Theorem 2.1. *The spectrum of A_0 consists of the union of the Cantor set $\mathbb{K}$ and a copy of the set of natural numbers $\mathbb{N}$. Each element of $\mathbb{N}$ is an isolated point and a fundamental system of neighborhoods of a point $\gamma = (\gamma_i)$ in $\mathbb{K}$ consists of the sets V_k defined to be the union of the sets*

$$\{\zeta \in \mathbb{K}: \zeta_i = \gamma_i, i < k\}$$

and

$$\{n \in \mathbb{N}: n \geq k \quad \text{and} \quad \beta_i^{(n)} = \gamma_i, i < k\}.$$

Note the interesting interplay between the digital representation of the natural numbers on one hand, and of elements of the Cantor set, on the other.

3. The Main Result

Recall from [**7**], Theorem 4.21, that a regular semi-saturated action of S^1 on a C^*-algebra, causes it to be isomorphic to the covariance algebra of a certain partial automorphism of the fixed point subalgebra. In the case of the action α of S^1 on A, described above, it is very easy to see that it is regular and semi-saturated. The semi-saturation follows immediately, since every weighted shift belongs to the first spectral subspace of α, henceforth denoted A_1. The fact that α is regular depends on the existence of an isomorphism $\theta: A_1^* A_1 \to A_1 A_1^*$ and a linear isometry $\lambda: A_1^* \to A_1 A_1^*$ such that, for x, $y \in A_1$, $a \in A_1^* A_1$ and $b \in A_1 A_1^*$

(i) $\lambda(x^* b) = \lambda(x^*) b$

(ii) $\lambda(a x^*) = \theta(a) \lambda(x^*)$

(iii) $\lambda(x^*)^* \lambda(y^*) = x y^*$

(iv) $\lambda(x^*) \lambda(y^*)^* = \theta(x^* y)$.

See [**7**], 4.3 and 4.4 for more information. It is easy to see that θ and λ, given by $\theta(a) = S a S^*$ and $\lambda(x^*) = S x^*$, satisfy the desired properties, where S denotes the unilateral (unweighted) shift.

Note that, in the present case, $A_1^* A_1 = A_0$ while $A_1 A_1^*$ is the ideal of A_0 formed by all sequences for which the first coordinate vanishes. That is, under the standard correspondence between ideals and open subsets of the spectrum, $A_1A_1^*$ corresponds to $X-\{0\}$. The isomorphism $\theta: C(X) \to C(X - \{0\})$ therefore induces a homeomorphism $f: X \to X - \{0\}$ which we would now like to describe.

If an integer n is thought of as an element of X, as seen above, then the element δ_n of l_∞, represented by the sequence having the n^{th} coordinate equal to one and zeros everywhere else, corresponds to the characteristic function of the singleton $\{n\}$ and, given that $\theta(\delta_n) = \delta_{n+1}$, we see that $f(n) = n + 1$.

Now, if $\gamma = (\gamma_i)$ is in $\mathbb{K}$, let $\gamma|_k$ be the k^{th} truncation of γ, that is $\gamma|_k = (\gamma_0, \dots, \gamma_{k-1})$ so that we can consider $e_{\gamma|_k}$, as defined above. Also let f_k be the element of c_0 given by $f_k = (1, \dots, 1, 0, \dots)$ where the last "1" occurs in the position $k-1$, counting from zero. The support of the Gelfand transform of the idempotent element $(1-f_k)e_{\gamma|_k}$ is precisely the set V_k referred to in 2.1. So, as k varies, the intersection of these sets is precisely the singleton $\{\gamma\}$. Therefore, to find out what $f(\gamma)$ should be, it is enough to look for the intersection of the supports of the Gelfand transforms of the elements

$$\theta((1 - f_k)e_{\gamma|_k}) = S(1 - f_k)e_{\gamma|_k}S^*.$$

The reader is now invited to verify, using this method, that the effect that f has on γ is precisely the effect of the odometer map. One should exercise special attention to check that the above method does indeed give $f(q_0 - 1, q_1 - 1, \dots) = (0, 0, \dots)$, in contrast with the partial automorphisms that produce UHF-algebras [8], since $(q_0 - 1, q_1 - 1, \dots)$ is removed from the domain of the maps considered there. Summarizing our findings so far, we have:

Theorem 3.1. *The Bunce-Deddens-Toeplitz C^*-algebra $A(\{n_k\})$ is isomorphic to the crossed product of $C(X)$ by the partial automorphism $\theta: C(X) \to C(X - \{0\})$ induced by the (inverse of the) map $f: X \to X - \{0\}$ acting like the odometer on $\mathbb{K}$ and like translation by one on $\mathbb{N}$.*

Recall that the Bunce-Deddens C^*-algebras were defined to be the quotient $B = A/\mathcal{K}$. If one identifies A with the crossed product above, it is easy to see that the ideal $\mathcal{K}$ corresponds to the crossed product of c_0 by the corresponding restriction of θ. Therefore, the quotient can be described as the (classical) crossed product of the Cantor set by the odometer map. That is:

Theorem 3.2. ([2], 10.11.4) *The Bunce-Deddens C^*-algebra $B(\{n_k\})$ is isomorphic to the crossed product of $C(\mathbb{K})$ by the automorphism induced by the odometer map.*

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